Optimistic Online Learning in Symmetric Cone Games

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Game Theory Formalism

- Finite number of players $\mathcal{N} := \{1, \cdots, N\}$.
- Strategy set of player i: $\mathcal{X}_i$, joint strategy set: $\mathcal{X} := \prod_{i=1}^{N} \mathcal{X}_i$.
- Utility $u_i : \mathcal{X} \to \mathbb{R}$ assumed to be concave w.r.t. its ith variable and differentiable.

ZOOM on the strategy spaces of agents (typically convex).

Examples of Games

- **Finite normal-form games** $\mathcal{X}_i = \Delta(\mathcal{A}_i)$: Golden standard.
- **Convex games with ball strategy sets** $\mathcal{X}_i = B(0, r_i)$.
- **PSD matrix games** $\mathcal{X}_i = \Delta_{\mathbb{S}_+^n}$.
  - Each player controls a PSD matrix variable (e.g. a signal covariance matrix) .
  - Applications in wireless communication networks for the competitive maximization of mutual information in interfering networks [Arslan et al., 2007, Scutari et al., 2008, Mertikopoulos and Moustakas, 2015, Majlesinasab et al., 2019].
- **Quantum games** $\mathcal{X}_i = \Delta_{\mathbb{H}_+^n}$.
  - Strategies of the players are quantum states represented by density matrices.
  - Utility is the expected value of a measurement on the joint state.

Example 1: Distance Metric Learning

- E.g. Learn a Mahalanobis distance given a dataset $\{x_i\}_{1\leq i\leq N}$ where $x_i \in \mathbb{R}^d$.
- [Ying and Li, 2012]

$$\begin{aligned} \max_{M\in\mathbb{S}^d_+} \quad & \min_{(i,j)\in\mathcal{D}} \underbrace{d^2_M(x_i,x_j)}_{(x_i-x_j)^T M(x_i-x_j)} \\ \text{s.t.} \quad & \sum_{(i,j)\in\mathcal{S}} d^2_M(x_i,x_j) \leq 1 \end{aligned} \tag{1}$$

Simplex-Spectraplex Game

$$\min_{x\in\Delta^{m-1}} \max_{Y\in\Delta_{\mathbb{S}^d_+}} f(x,Y) := \langle Y, \mathcal{A}(x)\rangle + \langle b, x\rangle + \langle C, Y\rangle \ , \tag{2}$$

where $\mathcal{A}:\mathbb{R}^m \to \mathbb{S}^d$ is the linear map given by $\mathcal{A}(x) = \sum_{i=1}^m x_i A_i$ for some $A_i \in \mathbb{S}^d$.

- Smoothing [Nesterov, 2007] in $\mathcal{O}(1/\varepsilon)$ iterations, interior point methods ...

Example 2: Fermat-Weber Problem

$$\min_{x \in B(0,R)} \left\{ g(x) := \sum_{i=1}^{p} \|A_i x - b_i\|_2 \right\}.$$

- **Min-max reformulation:** variational characterization of the maximal eigenvalue,

$$g(x) = \sum_{i=1}^{p} \lambda_{\max}(\bar{A}_i x - \bar{b}_i) = \sum_{i=1}^{p} \max_{\bar{u}_i \in \Delta_{\mathcal{K}}} \langle \bar{u}_i, \bar{A}_i x - \bar{b}_i \rangle_{\mathcal{J}} = \max_{y \in \Delta_{\mathcal{K}}^{p}} \langle y, \bar{A} x - \bar{b} \rangle_{\mathcal{J}^p}$$

where $\bar{A}_i := \left(0, A_i^T\right)^T \in \mathbb{R}^{(m+1) \times d}$ and $\bar{b}_i := \left(0, b_i^T\right)^T \in \mathbb{R}^{m+1}$, $\mathcal{J} = \mathbb{L}^{d+1}$.

Second-Order Cone Min-Max Game

$$\min_{\bar{x} = (1/2, \tilde{x}) \in \Delta_{\mathbb{L}_+^{d+1}}} \max_{y \in \prod_{i=1}^{p} \Delta_{\mathbb{L}_+^{d+1}}} \left\{ f(\bar{x}, y) := \langle \tilde{A} \bar{x}, y \rangle - \langle \bar{b}, y \rangle \right\}, \ \tilde{A} := 2R(0 \quad \bar{A}). \qquad (3)$$

- IP method of [Xue and Ye, 1997], extension of smoothing [Baes, 2006].

Questions

- Can we unify all these games?
- Can we learn Nash equilibria in the 2-player zero-sum setting using a **SINGLE ALGORITHM** for all these games simultaneously?

Two-player zero-sum SCGs

$$\min_{x \in \Delta_{\mathcal{K}_1}} \max_{y \in \Delta_{\mathcal{K}_2}} f(x, y)$$

SHORT ANSWERS

- Can we unify all these games?

SYMMETRIC CONE GAMES

- Can we learn Nash equilibria efficiently in the 2-player zero-sum setting using a single algorithm for all these games simultaneously?

OPTIMISTIC SYMMETRIC CONE MULTIPLICATIVE WEIGHTS

Outline

1. **Symmetric Cones and SC Games**

2. **Optimistic Online Learning in Symmetric Cone Games**

3. **Applications to Min-Max Problems over Symmetric Cones**

Two-player zero-sum SCGs

$$\min_{x \in \Delta_{\mathcal{K}_1}} \max_{y \in \Delta_{\mathcal{K}_2}} f(x, y)$$

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Examples of Symmetric Cones and corresponding EJAs

EJA $\mathcal{J}$	Inner product $\langle x, y \rangle$	Jordan product $x \circ y$	Cone of squares $\mathcal{K}$
Euclidean space $(\mathbb{R}^n)$	$\sum_{i=1}^n x_i y_i$	$(x_i y_i)_{i=1,\cdots,n}$	nonnegative orthant $(\mathbb{R}^n_+)$
Real sym. matrices $(\mathbb{S}^n)$	$\mathrm{tr}(xy)$	$\frac{1}{2}(xy + yx)$	PSD cone $(\mathbb{S}^n_+)$
Jordan spin algebra $(\mathbb{L}^n)$	$2\sum_{i=1}^n x_i y_i$	$(\bar{x}^\top \bar{y}, x_1 \bar{y} + y_1 \bar{x})$	second-order cone $(\mathbb{L}^n_+)$

- Characterization of SCs and formalism of Euclidean Jordan Algebras [Faraut and Korányi, 1994]: *Any* symmetric cone is the cone of squares $\{x \circ x : x \in \mathcal{J}\}$ of some EJA $\mathcal{J}$.

Strategy sets: Generalized Simplexes

Generalized Simplex

If $(\mathcal{J}, \circ)$ is an EJA and $\mathcal{K}$ its cone of squares,

$$\Delta_{\mathcal{K}} := \{x \in \mathcal{K} : \mathrm{tr}(x) = 1\}.$$

Symmetric Cone $\mathcal{K}$	Generalized Simplex $\Delta_{\mathcal{K}}$
Nonnegative orthant $\mathbb{R}^n_+$	Simplex $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$
Real PSD symmetric matrices $\mathbb{S}^n_+$	Spectraplex $\Delta_{\mathbb{S}^n_+} = \{X \in \mathbb{S}^n_+ : \mathrm{Tr}(X) = 1\}$
PSD Hermitian matrices $\mathbb{H}^n_+$	Spectraplex $\Delta_{\mathbb{H}^n_+} = \{X \in \mathbb{H}^n_+ : \mathrm{Tr}(X) = 1\}$
Second-order cone $\mathbb{L}^n_+$	Ball $\Delta_{\mathbb{L}^n_+} = \{(\frac{1}{2}, x) \in \mathbb{R}^n : \|x\|_2 \leq \frac{1}{2}\}$

Definition: Symmetric Cone Games

- Finite number of players $\mathcal{N} := \{1, \cdots, N\}$.
- Strategy set of player i: generalized simplex $\Delta_{\mathcal{K}_i}$ where $\mathcal{K}_i$ symmetric cone.
  - Notation: Space of joint strategies $\mathcal{X} := \prod_{i\in\mathcal{N}} \Delta_{\mathcal{K}_i}$.
- Utility $u_i : \mathcal{X} \to \mathbb{R}$ assumed to be concave w.r.t. its ith variable and differentiable.
  - Notation: Payoff vector,

$$\forall x \in \mathcal{X},\ m(x) = (m_i(x))_{i\in\mathcal{N}}, \quad m_i(x) = \nabla_{x_i} u_i(x_i, x_{-i}), \forall i \in \mathcal{N}.$$

Examples of Symmetric Cone Games

- Finite normal-form games
- Convex games with ball strategy sets
- PSD matrix games
- Quantum games

- Nonnegative orthant
- 2nd order cone
- Cone of PSD matrices
- Cone of complex PSD matrix

Online Learning in Symmetric-Cone Games

Initialization: $x_i^1 \in \Delta_{\mathcal{K}_i}, \forall i \in \mathcal{N}$
For $t = 1, \ldots, T$:
For $i \in \mathcal{N}$ (simultaneously):
Player i observes their (and only theirs) payoff $m_i^t \in \mathcal{J}_i$
Player i receives linear payoff $\langle m_i^t, x_i^t \rangle$
Player i computes their next strategy $x_{t+1}^i \in \Delta_{\mathcal{K}_i}$
Output: $\bar{x}_T^i = \frac{1}{T} \sum_{t=1}^T x_i^t \,, \forall i \in \mathcal{N}$.

*canonical EJA inner product $\langle x, y \rangle = tr(x \circ y)$.

Playerwise Regret

$$r_i(T) := \sup_{x \in \Delta_{\mathcal{K}_i}} \sum_{t=1}^{T} u_i^t(x, x_{-i}^t) - u_i(x_i^t, x_{-i}^t) \,, \quad \forall i \in \mathcal{N} .$$

Optimistic Follow The Regularized Leader

- Fix player $i \in \mathcal{N}$.

OFTRL

$$x^{t+1} = \underset{x \in \Delta_{\mathcal{K}}}{\operatorname{argmax}} \left\{ \underbrace{\eta}_{\text{step size}} \left\langle \sum_{k=1}^{t} m^k + \tilde{m}^{t+1}, x \right\rangle - \underbrace{\Phi(x)}_{\text{SC regularizer}} \right\}$$

where $m^k = \nabla_{x_i} u_i(x_i^k, x_{-i}^k)$ and $(\tilde{m}^t)$ is a predictor sequence, typically $\tilde{m}^{t+1} = m^t$.

- Why optimism?

- Which regularizer?

Why Optimistic Online Learning? Related work

- In min-max problems (under min-max theorem):
  - average regret is bounded above by $\varepsilon \implies$ time-average iterate is an ε-saddle point.

Fully adversarial setting	Game setting
Average regret	Average regret in 2pZS and N-player NF
$\mathcal{O}(T^{-1/2})$	$\tilde{\mathcal{O}}(T^{-1})$

- [Daskalakis et al., 2011, Chiang et al., 2012, Rakhlin and Sridharan, 2013, Syrgkanis et al., 2015, Daskalakis et al., 2021] ...
- [Vasconcelos et al., 2023] zero-sum quantum games, optimization perspective.

Regularizer? Symmetric Cone Negative Entropy

Negative Entropy

Given the EJA $\mathcal{J}$ and its cone of squares $\mathcal{K}$, $\Phi_{\text{ent}} : \text{int}(\mathcal{K}) \to \mathbb{R}$:

$$\forall x \in \text{int}(\mathcal{K}), \quad \Phi_{\text{ent}}(x) = \text{tr}(x \circ \ln x) = \sum_{i=1}^{r} \lambda_i \ln \lambda_i \,, \tag{SCNE}$$

- $x = \sum_{i=1}^{r} \lambda_i q_i \in \text{int}(\mathcal{K})$ spectral decomposition of x.
- $\ln : \text{int}(\mathcal{K}) \to \mathcal{J}$ Löwner extension of the scalar log, i.e. $\ln x = \sum_{i=1}^{r} \ln(\lambda_i) q_i$.
- Note that $\lambda_i > 0$ for every $1 \leq i \leq r$ as $x \in \text{int}(\mathcal{K})$.

- Exponential mapping $\exp : \mathcal{J} \to \text{int}(\mathcal{K})$ is defined by $\exp(x) = \sum_{i=1}^{r} \exp(\lambda_i)\, q_i$.

Strong Convexity of Symmetric Cone Negative Entropy

Theorem

Let $(\mathcal{J}, \circ)$ be an EJA and let $\mathcal{K}$ be its cone of squares. Then,

$$\forall x, y \in \operatorname{int}(\Delta_{\mathcal{K}}), \quad D_{\Phi_{\text{ent}}}(x, y) \geq \frac{1}{2}\|x - y\|_{tr,1}^2 . \tag{4}$$

where $D_{\Phi_{\text{ent}}}$ is the Bregman divergence $D_{\Phi_{\text{ent}}}(x, y) = \operatorname{tr}(x \circ \ln x - x \circ \ln y + y - x)$.

- Proof sketch:

$$D_{\Phi}(x, y) \underset{(1+2)}{\geq} D_{\Phi}(T(x), T(y)) = \mathrm{KL}(u(x)||u(y)) \geq \|u(x) - u(y)\|_1^2 = \frac{1}{2}\|x - y\|_{tr,1}^2 ,$$

1. The diagonal mapping is a convex combination of EJA automorphisms
2. Use joint convexity of the relative entropy and properties of automorphisms.

- Alternative proof [Baes, 2006] (Hessian and duality).

Optimistic Symmetric Cone Multiplicative Weights Update

- Regularizer Φ = Symmetric Cone Negative Entropy

OSCMWU Algorithm

$$w^{t+1} = \eta \left(\sum_{k=1}^{t} m^k + \tilde{m}^{t+1} \right), \qquad x^{t+1} = \frac{\exp(w^{t+1})}{\mathrm{tr}(\exp(w^{t+1}))}, \quad \forall t \geq 1,$$

where $(\tilde{m}^t)$ is a predictor sequence, typically $\tilde{m}^{t+1} = m^t$.

- $\tilde{m}^{t+1} = 0$: SCMWU introduced and studied recently in [Canyakmaz et al., 2023]

Regret in Symmetric Cone Games

Theorem [Syrgkanis et al., 2015]

Under smoothness of payoff vectors, if each player $i \in \mathcal{N}$ runs OSCMWU for T rounds on $\Delta_{\mathcal{K}_i}$ with stepsize $\eta = 1/(2\sqrt{N\sum_{i=1}^{N} L_i^2})$ and set $\|\cdot\| = \|\cdot\|_{tr,1}$. Then

$$\sum_{i=1}^{N} r_i(T) \leq 2\left(\sum_{i=1}^{N} R_i\right) \cdot \sqrt{N\sum_{i=1}^{N} L_i^2}\,, \tag{5}$$

where $r_i(T)$ is the i-th player's regret and $R_i = \sup_{x\in\Delta_{\mathcal{K}_i}} \Phi(x) - \inf_{x\in\Delta_{\mathcal{K}_i}} \Phi(x)$.

Application to 2-Player Zero-Sum Symmetric Cone Games

Min-Max Problem

$$\min_{x \in \Delta_{\mathcal{K}_1}} \max_{y \in \Delta_{\mathcal{K}_2}} f(x, y),$$

- $f : \mathcal{J}_1 \times \mathcal{J}_2 \to \mathbb{R}$ is convex-concave and differentiable.
- $\mathcal{K}_1, \mathcal{K}_2$ are arbitrary symmetric cones and $\Delta_{\mathcal{K}_1}, \Delta_{\mathcal{K}_2}$ their generalized simplexes.

Theorem (2-player Zero-Sum SCG) - adaptation of folklore result

If both players run OSCMWU with stepsize $\eta = 1/(2\sqrt{2(L_1^2 + L_2^2)})$,

$$T \geq \frac{2(\ln r_1 + \ln r_2)\sqrt{2(L_1^2 + L_2^2)}}{\varepsilon} \implies \left(\bar{x}_T = \frac{1}{T}\sum_{t=1}^{T} x^t, \bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y^t\right) \quad \varepsilon\text{-SP}.$$

$r_i = \operatorname{rank}(\mathcal{J}_i), i = 1, 2.$

Conclusion

- **SCGs:** Exploit geometric structure and unify several existing problems.
  - Normal form, quantum, PSD games, convex games with ball strategy sets.
- **OSCMWU:** single algorithm applying in a unified way instead of there exist ad-hoc algorithms for special cases of our setting.
  - **Log dependence on the intrinsic dimension of the problem**.
- **Applications** beyond normal-form and quantum games.
  - simplex-spectraplex, second-order cone games.

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Symmetric Cones and Euclidean Jordan Algebras I

Definition of symmetric cones

- A cone $\mathcal{K}$ in an inner product space is *symmetric* if self-dual and homogeneous.

Let $\mathcal{J}$ be a finite dim. vector space with a bilinear product $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$.

- $(\mathcal{J}, \circ)$ Jordan algebra if:

$$\forall x, y \in \mathcal{J}, \quad x \circ y = y \circ x\,, \quad \underbrace{x^2}_{x \circ x} \circ (x \circ y) = x \circ (x^2 \circ y)\,.$$

- $(\mathcal{J}, \circ, (\cdot, \cdot))$: *Euclidean* Jordan algebra over $\mathbb{R}$ if equipped with an associative inner product $(\cdot, \cdot)$, i.e. for all $x, y, z \in \mathcal{J}, (x \circ y, z) = (y, x \circ z)\,.$

Characterization of Symmetric cones

- *Any* symmetric cone is the cone of squares $\{x \circ x : x \in \mathcal{J}\}$ of some EJA $\mathcal{J}$ [Faraut and Korányi, 1994]

A Simplex-Spectraplex Game (2/2) I

- **Iteration complexity of OSCMWU:** $T \geq \frac{4(\ln m + \ln d) \max_i \|A_i\|_{\mathrm{tr},\infty}}{\varepsilon}$ iterations.
- **Prior work:**
  - Smoothing technique [Nesterov, 2007] in $\mathcal{O}(1/\varepsilon)$ iterations (similar).
  - Frank-Wolfe method + smoothing [Ying and Li, 2012] in $\mathcal{O}(1/\varepsilon^2)$ iterations but only $\mathcal{O}(d^2)$ runtime cost.
  - Interior point methods: high precision but prohibitive per-iteration cost for large-scale problems.

A 2nd-Order Cone Min-Max Game: (2/2) I

$$\min_{\bar{x}=(1/2,\tilde{x})\in\Delta_{\mathbb{L}_+^{d+1}}} \max_{y\in\prod_{i=1}^{p}\Delta_{\mathbb{L}_+^{d+1}}} \left\{ f(\bar{x},y) := \langle \tilde{A}\,\bar{x}, y\rangle - \langle \bar{b}, y\rangle \right\}, \; \tilde{A} := 2R(0 \quad \bar{A})\,.$$

- **Iteration complexity of OSCMWU:** $T \geq \frac{4(p+1)L\ln 2}{\varepsilon}$ iterations.
- **Prior work:**
  - Extension of the smoothing technique of Nesterov to EJAs [Baes, 2006].
  - Interior point method of [Xue and Ye, 1997] requires fewer iterations but much higher per-iteration cost.

Regret Bounded by Variation in Utilities [Syrgkanis et al., 2015] I

RVU

For (x^t) generated by OFTRL with $\tilde{m}^{t+1} = m^t$ and a regularizer Φ that is 1-strongly convex w.r.t. a norm $\|\cdot\|$, for all $T \geq 1$,

$$\forall x \in \Delta_{\mathcal{K}}, \quad \sum_{t=1}^{T} f^t(x) - f^t(x^t) \leq \frac{R}{\eta} + \eta \sum_{t=1}^{T} \|m^t - m^{t-1}\|_*^2 - \frac{1}{4\eta} \sum_{t=1}^{T} \|x^t - x^{t-1}\|^2 ,$$

where $R = \sup_{x \in \Delta_{\mathcal{K}}} \Phi(x) - \inf_{x \in \Delta_{\mathcal{K}}} \Phi(x)$, $\|\cdot\|_*$ is the dual norm and $\langle \cdot, \cdot \rangle$ the EJA inner product.

Online Symmetric-Cone Optimization (OSCO)

Initialization: $x^1 \in \Delta_{\mathcal{K}} := \{x \in \mathcal{K} : \mathrm{tr}(x) = 1\}$

For $t = 1, \ldots, T$:

Observe the payoff $m^t \in \mathcal{J}$

Receive linear payoff $\langle m^t, x^t \rangle$

Compute new iterate $x^{t+1} \in \Delta_{\mathcal{K}}$